6558/KA3 **OCTOBER 2007**

Paper III — TOPOLOGY AND MEASURE THEORY

(For those who joined in July 2003 and after)

Time : Three hours Maximum : 100 marks

SECTION A — (4 × 10 = 40 marks)

Answer any FOUR questions.

1. If Y is a subspafce of X, show that a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.

2. If X and Y are topological spaces and $f: X \to Y$, prove that the following are euquivalent :

(a) f is continuous.

(b) For every subset A of X, one has $f(\overline{A}) \subset \overline{f(A)}$.

(c) For every closed set B in Y, the set $f^{-1}(B)$ is closed in X.

3. Show that every compact subset of a Hausdorff space is closed.

4. Prove that every metrizable space is normal.

5. Show that the outer measure of an interval is its length.

6. State and prove Lebesgue convergence theorem.

7. State and prove Hahn decomposition theorem.

8. Define an outer measure u^*. Show that the class of u^*–measurable sets is a σ – algebra.

SECTION B — $(3 \times 20 = 60$ marks)

Answer any THREE questions.

9. (a) Define product topology. If A is a subspace of X and B is a subspace of Y, show that the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

(b) If X is a Hausdorff space and A is a subset of X, prove that the point x is a limit point of A if and only if every neighbourhood of x contains infinitely many points of A.

10. Show that the product of finitely many compact spaces is compact.

11. State and prove Urysohn's lemma.

12. (a) State and prove bounded convergence theorem for measurable functions.

(b) State and prove Fatou's lemma.

13. State and prove Radon-Nikodym theorem.

14. (a) State and prove Lebesgue decomposition theorem for measures.

(b) State and prove Fubini's theorem.